APPLICATION OF A DIFFERENTIAL-DIFFERENCE METHOD TO THE SOLUTION OF A ONE-DIMENSIONAL NONSTATIONARY TWO-LAYER HEAT CONDUCTION PROBLEM WITH A MOVING BOUNDARY

L. Ya. Zhemoidina

UDC 536.248.2

We apply a differential-difference method to obtain an approximate solution of a nonstationary heat conduction problem with a moving boundary for a medium consisting of an unbounded plate ($0 \leq x \leq l$) and a halfspace ($l < x < \infty$), possessing various thermophysical properties.

Statement of the Problem and Its Solution. In many of the problems of thermotechnics one is concerned with heat conduction problems in which the heat exchange is accompanied by a phase transformation. As examples, we cite problems of melting and hardening of solids. The essential feature of these problems is the presence of a boundary separating the phases, whose displacement depends on the time.

For sufficiently strong heat sources the material at the surface melts, vaporizes, and is carried away by the external flow. In this case the problem is one involving a single phase.

We consider a problem of this kind for the case of a two-layer medium, consisting of an unbounded plate ($0 \leq x \leq l$) and a halfspace ($l < x < \infty$) with various thermophysical properties, the temperature being identical at all points of each plane x = const. The temperature of such a medium then satisfies the one-dimensional heat conduction equation. We assume that the plate surface is heated by a constant heat source q, and that initially the temperature of the whole medium is a constant, which we take equal to zero.

Up to the instant that the boundary begins to move, this problem is a two-layer heat conduction problem, one which, for given conditions, was considered in [1] and [2], where with the aid of the Laplace transformation the temperature distribution was obtained in each medium.

Commencing with this solution as our initial stage (i.e., up to the instant of melting), we can find a time, which defines the start of melting, and, consequently, also the instant at which the boundary begins to move. Moreover we can take as our reference origin the instant at which melting commences. The problem so stated may be formulated mathematically as follows: from the equations

$$c_1\rho_1 \frac{\partial u_1}{\partial t} = k_1 \frac{\partial^2 u_1}{\partial x^2}, \quad \xi(t) < x < l, \quad 0 < t < T, \tag{1}$$

$$c_2\rho_2 \frac{\partial u_2}{\partial t} = k_2 \frac{\partial^2 u_2}{\partial x^2}, \quad l < x < \infty, \quad 0 < t < T \tag{2}$$

we wish to find temperatures $u_1(x, t)$ and $u_2(x, t)$, satisfying the initial conditions

$$\begin{gathered} u_1(x, t)|_{t=0} = \varphi(x), \quad 0 \leqslant x < l, \quad \varphi(0) = u_{\mathrm{m}}, \\ u_2(x, t)|_{t=0} = \psi(x), \quad l < x < \infty \end{gathered} \tag{3}$$

and the boundary conditions

$$u_1(x, t)|_{t=\xi(t)} = u_{\mathrm{m}}, \quad u_2(x, t)|_{x=\infty} = 0, \tag{4}$$

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 19, No. 6, pp. 1026-1030, December, 1970. Original article submitted August 8, 1969.

and also the conditions of continuity of temperature and heat flow at the interface

$$u_1(l, t) = u_2(l, t), \quad k_1 \frac{\partial u_1}{\partial x}\bigg|_{x=l} = k_2 \frac{\partial u_2}{\partial x}\bigg|_{x=l}. \tag{5}$$

Here $\varphi(x)$ and $\psi(x)$ are the temperature distributions at the instant melting begins, obtained from the solution to the initial stage of the problem.

To find the position of the melting front $x = \xi(t)$, we use the condition which is satisfied at the phase transition surface:

$$\lambda\rho_1 \frac{d\xi}{dt} = k_1 \frac{\partial u_1}{\partial x}\bigg|_{x=\xi(t)} + q, \quad 0 < t < T. \tag{6}$$

Condition (6) makes sense only for $0 < x \leq l$, since for $x > l$ the problem becomes a single-layer problem, a solution for which is given in [3].

We solve the problem by applying the following differential-difference scheme.

In Eqs. (1) and (2) we put $t = t_{m+1}$, and we replace the derivative with respect to t by the finite-difference ratios

$$\frac{\partial u_1(x, t)}{\partial t}\bigg|_{t=t_{m+1}} \approx \frac{u_1(x, t_{m+1}) - u_1(x, t_m)}{h}, \tag{7}$$

$$\frac{\partial u_2(x, t)}{\partial t}\bigg|_{t=t_{m+1}} \approx \frac{u_2(x, t_{m+1}) - u_2(x, t_m)}{h}, \tag{8}$$

where $h = t_{m+1} - t_m$ is the time-step.

Substituting the expressions (7) and (8), respectively, into Eqs. (1) and (2), we obtain a system of ordinary differential equations

$$\frac{d^2u_{1,m+1}(x)}{dx^2} - a_1^2 u_{1,m+1}(x) = -a_1^2 u_{1,m}(x), \quad \xi(t_m) < x < l, \tag{9}$$

$$\frac{d^2u_{2,m+1}(x)}{dx^2} - a_2^2 u_{2,m+1}(x) = -a_2^2 u_{2,m}(x), \quad l < x < \infty \tag{10}$$

$$(m = 0, 1, 2, \ldots).$$

Here $a_i^2 = c_i\rho_i/k_i h$ and the $u_{i,m+1}(x)$ approximate $u_i(x, t_{m+1})$ $(i = 1, 2)$.

The initial and boundary conditions are transformed as follows:

$$\begin{gathered} u_1(x, t)|_{t=0} = u_{1,0}(x) = \varphi(x), \quad 0 \leqslant x < l, \quad \varphi(0) = u_m, \\ u_2(x, t)|_{t=0} = u_{2,0}(x) = \psi(x), \quad l < x < \infty, \\ u_{1,m+1}(x)|_{x=\xi(t_{m+1})} = u_m, \quad u_{2,m+1}(x)|_{x=\infty} = 0, \\ u_{1,m+1}(l) = u_{2,m+1}(l), \quad k_1 \frac{du_{1,m+1}(x)}{dx}\bigg|_{x=l} = k_2 \frac{du_{2,m+1}(x)}{dx}\bigg|_{x=l} \\ (m = 0, 1, 2 \ldots). \end{gathered} \tag{11}$$

Similarly, by replacing the time derivative $d\xi(t)/dt$ by the finite-difference ratio $[\xi(t_{m+1}) - \xi(t_m)]/h$ in Eq. (6), we obtain equations for determining the ξ_{m+1}, approximating the values $\xi(t_{m+1})$

$$\xi_{m+1} = \xi_m + \frac{c_1}{a_1^2\lambda}\left[\frac{du_{1,m+1}(x)}{dx}\bigg|_{x=\xi_{m+1}} + \frac{q}{k_1}\right] \tag{12}$$

$$(m = 0, 1, 2, \ldots).$$

The solutions of Eqs. (9) and (10), obtained by the method of variation of parameters, can, upon taking into account the conditions (11) and the approximate equality $u_{1,m+1}[\xi(t_{m+1})] \approx u_{1,m+1}(\xi_{m+1})$, be put into the form

$$u_{1,m+1}(x)=\frac{\operatorname{sh} a_1(x-\xi_{m+1})}{P(\xi_{m+1})}\left\{a_1\int_0^l u_{1,m}(z)P(z)\,dz+a_2^2k_2\int_l^\infty u_{2,m}(z)\exp\left[-a_2(z-l)\right]dz\right\}$$

$$+\frac{P(x)}{P(\xi_{m+1})}\left[a_1\int_0^{\xi_{m+1}} u_{1,m}(z)\operatorname{sh} a_1(\xi_{m+1}-z)\,dz+u_m\right]-a_1\int_0^x u_{1,m}(z)\operatorname{sh} a_1(x-z)\,dz, \tag{13}$$

$$u_{2,m+1}(x)=\frac{\exp\left[-a_2(x-l)\right]}{P(\xi_{m+1})}\left\{a_1^2k_1\int_0^l u_{1,m}(z)\operatorname{sh} a_1(z-\xi_{m+1})\,dz\right.$$

$$\left.+a_1k_1\left[a_1\int_0^{\xi_{m+1}} u_{1,m}(z)\operatorname{sh} a_1(\xi_{m+1}-z)\,dz+u_m\right]+a_2\int_l^\infty u_{2,m}(z)R(z,\ \xi_{m+1})\,dz\right\}-a_2\int_x^\infty u_{2,m}(z)\operatorname{sh} a_2(z-x)\,dz \tag{14}$$

$$(m=0,\ 1,\ 2,\ \ldots),$$

where

$$P(x)=a_1k_1\operatorname{ch} a_1(l-x)+a_2k_2\operatorname{sh} a_1(l-x),$$

$$R(z,\ x)=a_1k_1\operatorname{sh} a_2(z-l)\operatorname{ch} a_1(l-x)+a_2k_2\operatorname{ch} a_2(z-l)\operatorname{sh} a_1(l-x).$$

Expressions (13) and (14) can serve as recursion relations for determining the successive values of $u_{1,m+1}(x)$ and $u_{2,m+1}(x)$.

To find the values of the ξ_{m+1} we use Eqs. (12), which, after substitutions are made for $u'_{1,m+1}(\xi_{m+1})$, assume the form

$$\xi_{m+1}=\xi_m+\frac{c_1}{\lambda}\left\{\frac{1}{P(\xi_{m+1})}\left[\int_{\xi_{m+1}}^l u_{1,m}(z)P(z)\,dz\right.\right.$$

$$\left.\left.+\frac{a_2^2k_2}{a_1}\int_l^\infty u_{2,m}(z)\exp\left[-a_2(z-l)\right]dz+\frac{P'(\xi_{m+1})}{a_1^2}u_m\right]+\frac{q}{a_1^2k_1}\right\} \tag{15}$$

$$(m=0,\ 1,\ 2,\ \ldots).$$

We determine the displacement of the moving boundary for each time interval h.

We let

$$\xi_{m+1}-\xi_m=\delta_{m+1}$$

and

$$\Phi(\xi_{m+1})=\frac{1}{P(\xi_{m+1})}\left[\int_{\xi_{m+1}}^l u_{1,m}(z)P(z)\,dz+\frac{a_2^2k_2}{a_1}\int_l^\infty u_{2,m}(z)\exp\left[-a_2(z-l)\right]dz+\frac{P'(\xi_{m+1})}{a_1^2}u_m\right].$$

Then

$$\xi_{m+1}=\xi_m+\delta_{m+1} \tag{16}$$

and from Eq. (15) we obtain

$$\delta_{m+1}=\frac{c_1}{\lambda}\left[\Phi(\xi_m+\delta_{m+1})+\frac{q}{a_1^2k_1}\right]\quad(m=0,\ 1,\ 2,\ \ldots). \tag{17}$$

To determine δ_{m+1} we expand the function $\Phi(\xi_m+\delta_{m+1})$ in a Taylor series

$$\Phi(\xi_m+\delta_{m+1})=\Phi(\xi_m)+\delta_{m+1}\Phi'(\xi_m)+O(\delta_{m+1}^2).$$

Taking into account terms up to order δ_{m+1}^2, we obtain, from Eqs. (17),

$$\delta_{m+1}\approx\frac{\Phi(\xi_m)+\dfrac{q}{a_1^2k_1}}{\dfrac{\lambda}{c_1}-\Phi'(\xi_m)}\quad(m=0,\ 1,\ 2,\ \ldots), \tag{18}$$

where

$$\Phi'(\xi_m) = -\frac{P'(\xi_m)}{P(\xi_m)}\Phi(\xi_m).$$

Knowing δ_{m+1} from Eqs. (18), we can successively determine ξ_{m+1} for m = 0, 1, 2, . . . , since $\xi_0 = 0$ is known.

Using Rothe's lemmas [4, 5], we can prove convergence of the approximate solutions to the exact solutions and thus estimate the error of our method. In so doing, we can show that the errors made are of order O(h).

NOTATION

$u_1(x, t)$, $u_2(x, t)$	are the temperatures;
u_M	is the melting point;
$\xi(t)$	is the function determining the position of moving boundary;
c_1, c_2	are the specific heat fluxes;
ρ_1, ρ_2	are the density;
k_1, k_2	are the heat-transfer coefficients;
λ	is the latent heat of evaporation;
T	is the time of process under consideration.

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